ASYMPTOTIC SOLUTION OF THE PROBLEM OF OUTFLOW OF HEAVY LAMINAR JETS OF NONMIXING LIQUIDS

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UDC 532.536

The solution of problems concerned with the flow of heavy jets of ideal liquid is given in a number of publications (see, for example, [1-3]). The paper [4] is devoted to the asymptotic behavior of viscous jets without taking into account the surrounding medium and mass forces. Finally, viscous flows of nonmixing liquids without mass forces are considered in [5], where an approximate solution for a plane jet is obtained by the integral method.

We shall consider the problem of outflow of a vertical laminar jet into a different medium which does not mix with the outflowing liquid. The solution is carried out for the simplest formulation of this problem. We assume that the jet, over the entire extent of its flow, does not break and remains laminar; in addition, we assume that between the outflowing liquid and the medium there exists a smooth boundary (Fig. 1). As a consequence of friction, the outflowing liquid draws into motion the external medium adjacent to the jet; as a result, an associated mass is formed. Neglecting the narrow diffusive layer, we shall assume that on the separation boundary the condition of equal velocities and shear and normal stresses is fulfilled. The flow diagram adopted, although possessing a number of shortcomings, is nevertheless realized in practice. Thus, in the case of flow of a jet of dense liquid in air, under certain conditions there exists a fairly large part of the jet on which the separation boundary is a smooth surface.

An asymptotic method, allowing us to calculate the jet flow far from the source, is proposed for the solution of the problem formulated. Here the influence of the initial impulse and the initial velocity profile is not taken into account. The approach proposed is analogous to the analysis of jets of mixing liquids issuing from point sources [6].

In the role of the basic equations we use a system of Navier–Stokes equations in the region 1, which we call the internal region, and equations of the boundary layer in the region 2, which we call the external region. Such a difference in the mathematical description of the regions can be explained by the fact that the flow in region 1 constitutes in a certain way a flow in a channel with distorted walls. Therefore, here transverse pressure gradients, commensurable with the longitudinal gradients, are possible. In the region 2, however, the magnitudes of transverse pressure gradients are considerably less than the longitudinal gradients.

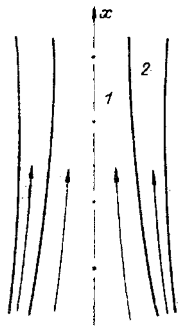

Fig. 1

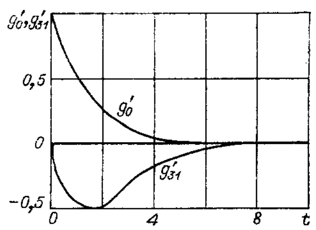

Fig. 2

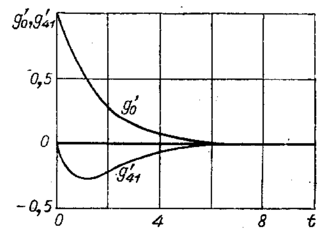

Fig. 3

Dnepropetrovsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 78-84, March-April, 1977. Original article submitted April 9, 1976.

1. Plane Jet. We write out the basic equations: in the region 1

$$u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\rho_1 \partial x} + \nu_1 \left\{ \frac{\partial^2 u_1}{\partial x^2} + \frac{1}{y^s} \frac{\partial}{\partial y} \left(y^s \frac{\partial u_1}{\partial y} \right) \right\} - g,$$

$$u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\rho_1 \partial y} + \nu_1 \left\{ \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial}{\partial y} \left(\frac{1}{y^s} \frac{\partial v_1}{\partial y} \right) \right\}, \tag{1.1}$$

$$\frac{\partial y^s u_1}{\partial x} + \frac{\partial y^s v_1}{\partial y} = 0,$$

where $s = 0$ for a plane problem and $s = 1$ for an axisymmetric problem; in the region 2

$$u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} = \nu_2 \frac{\partial^2 u_2}{\partial y^2}, \ \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

To obtain the solutions we determine the first terms in the expansions of the stream functions. Let

$$\psi_1 \sim a_0 U X^r F(n), \ n \sim y X^m / a_0;$$
$$\psi_2 \sim a_0 U X^k G(h), \ h \sim (y - a_0 X^{-m}) / a_0 X^p;$$
$$p_1 \sim -\rho_2 g x, \ \rho_2 > \rho_1,$$

where ψ is the stream function; $X = A \frac{\nu_1}{U a_0^2} x$; A is a certain constant quantity; U is the velocity scale; a_0 is the linear scale; and r, m, k and p are constant coefficients. From a condition of conservation of mass in the internal region we have $r = 0$. For the finding of the remaining coefficients we have the following conditions: a) the condition of conservation, in the equation for G in the external region, of the dynamic and viscous terms, i.e., $k + p = 1$; b) the condition of equality of the velocities on the separation boundary $m = k - p$; c) the condition of equality of the impulse increment of the associated mass $\frac{d}{dx} \rho_2 \int\limits_{y^*}^{\infty} u_2^2 dy$ to the resulting force acting on the issuing liquid $(\rho_2 - \rho_1) g y_*$, where y_* is the half-width of the internal region $(y_* \sim a_0 X^{-m})$. The first condition points to the equal importance of the viscous and dynamic terms in the external boundary layer. The last condition indicates that the Archimedean force applied to the jet proper is transmitted through the free surface to the associated mass, increasing its impulse. From these three conditions we can find $m = 1/5$, $k = 3/5$, $p = 2/5$. Thus, we can seek the solutions in the regions 1 and 2 in the form

$$\psi_1 = a_0 U [F_0(n) + X^{-1/5} F_1(n) + \dots], \tag{1.2}$$
$$n = y X^{1/5} / (a_0 + a_1 X^{-1/5} + a_2 X^{-2/5} + \dots);$$
$$\psi_2 = a_0 U \, X^{3/5} [G_0(h) + X^{-1/5} G_1(h) + \dots], \tag{1.3}$$
$$h = \frac{y - X^{-1/5} (a_0 + a_1 X^{-1/5} + \dots)}{X^{2/5} (a_0 + a_1 X^{-1/5} + \dots)}.$$

In the expressions (1.2) and (1.3), a_j $(j > 0)$ are corrective quantities. In order to avoid logarithmic terms in the solutions, we have to put $a_4 = 0$. In a contrary case, from the condition c) we see that the impulse of the associated mass, in addition to the power terms, will have a logarithmic term.

The boundary conditions on the axis of the jet and at infinity are given by the expressions

$$F_j(0) = 0, \ F_j''(0) = 0, \ G_j'(\infty) = 0.$$

On the separation boundary $(n = 1)$ the conditions of equality of the velocities and the shear stresses for the number of approximations considered are expressed in the form

$$G_j'(0) = F_j'(1), \ G_j''(0) = \gamma F_{j+3}''(1), \ \gamma = \frac{\mu_1}{\mu_2}. \tag{1.4}$$

Equations (1.4) are joining conditions of the solutions in the internal and external regions. Substituting (1.2) into Eqs. (1.1), we obtain the following system:

$$F_0''' = 0, \ F_1''' = 0, \ F_2''' = 0, \ F_3''' = -B, \ F_4''' = -3 \frac{a_1}{a_0} B,$$

$$F_5''' = -3 \left(\frac{a_2}{a_0} + \frac{a_1^2}{a_0^2} \right) B, \ F_6''' = -3 \left(\frac{a_3}{a_0} + 2 \frac{a_1 a_2}{a_0^2} + \right.$$

$$\left. + \frac{a_1^3}{a_0^3} \right) B + \frac{1}{5} A, \ F_7 = -3 \left(\frac{a_1 a_3}{a_0^2} + \frac{a_2^2}{a_0^2} + \frac{a_1^2 a_2}{a_0^3} \right) B + \frac{2}{5} \frac{a_1}{a_0} A,$$

where $B = g\frac{a_0^2}{U\nu_1}\frac{\rho_2-\rho_1}{\rho_1}$. Taking into account the fact that on the free surface

$$F_0(1) = 1,\ F_j(1) = 0\ (j > 0),$$

and taking into consideration the boundary conditions on the axis of the jet, we can write the solution in the form

$$\begin{gathered} F_0 = n,\ F_1 = 0,\ F_2=0,\ F_3 = (B/6)n(1-n^2),\ F_4 = 0,\ F_5 = 0, \\ F_6 = (1/6)(3(a_3/a_0)B - (1/5)A)n(1-n^2),\ F_7 = 0. \end{gathered} \tag{1.5}$$

In the expressions (1.5) the solutions are written out, with the fact taken into account that $a_1=a_2=0$. The last equation follows from the analysis of the solutions in the internal and external regions.

In the region 2, having replaced

$$G_j = Cg_j(t),\ t = Dh,$$

where $C=(\varkappa A)^{-1/2}$, $D=(\varkappa A)^{1/2}$; $\varkappa=\nu_1/\nu_2$, we obtain the following equations and boundary conditions:

$$g_0''' - \frac{1}{5}\left(g_0'^2 - 3g_0g_0''\right) = 0, \tag{1.6}$$

$$g_0(0) = 0,\ g_0'(0) = 1,\ g_0'(\infty) = 0,\ g_0''(0) = -\gamma D^{-1}B;$$

$$g_3''' + \frac{1}{5}\left(g_0'g_3' + 3g_0g_3''\right) = \frac{1}{5}\frac{a_3}{a_0}\left(4g_0'^2 - 3g_0g_0''\right), \tag{1.7}$$

$$g_3(0) = C^{-1},\ g_3'(0) = -\frac{B}{3},\ g_3'(\infty) = 0,\ g_3''(0) = \gamma D^{-1}\left(\frac{A}{5} - 3\frac{a_3}{a_0}B\right).$$

The boundary conditions just written out ensure the continuity of the stream functions, velocities, and shear stresses on the separation boundary. In the expressions (1.6), (1.7) we have not written out the equations for g_1, g_2 and g_4, since the solutions of these equations are trivial: $g_1=g_2=g_4=0$. The solution of Eq. (1.6), found numerically, is shown in Fig. 2. From the last boundary condition for $g_0''(0)$ we can find

$$A = \varkappa^{-1}\gamma^2B^2/(0.594)^2.$$

The solution of Eq. (1.7) can be represented in the form

$$g_3 = C_3g_0' + D_3 + \frac{a_3}{a_0}g_{31}.$$

Having determined numerically g_{31} for the boundary conditions (see Fig. 2), $g_{31}(0)=0$, $g'_{31}(0)=0$ and $g'_{31}(\infty)=0$, we find

$$C_3 = 0.561\,B,\ D_3 = C^{-1} - C_3,\ a_3/a_0 = -0.362(1/3 - \varkappa^{-1}\gamma^2)B.$$

We now write out the impulse for the entire jet:

$$I = \int_0^{y_*}\rho_1u_1^2dy + \int_{y_*}^{\infty}\rho_2u_2^2dy.$$

Substituting here the solutions just found, we have

$$I = \rho_2U^2a_0X^{4/5}\left[\int_0^{\infty}G_0'^2\,dh + X^{-3/5}\int_0^{\infty}\left(2G_0'G_3' - \frac{a_3}{a_0}G_0'^2\right)dh\right] + \rho_1U^2a_0X^{1/5}.$$

The magnitude of this impulse is equal to the resultant force acting on the outflowing liquid:

$$I = \rho_1B(U^2/A)a_0[(5/4)X^{4/5} + 5(a_3/a_0)X^{1/5}].$$

Having put $B=1$, from the condition of conservation of mass

$$G = \int_0^{y_*}\rho_1u_1dy = \text{const}$$

we find U and a_0:

$$U = \left[g\frac{G^2}{\nu_1\rho_1^2}\frac{\rho_2-\rho_1}{\rho_1}\right]^{1/3},\ a_0 = \frac{G}{\rho_1}\left[g\frac{G^2}{\nu_1\rho_1^2}\frac{\rho_2-\rho_1}{\rho_1}\right]^{-1/3}.$$

The equation B=1 was assumed for the sake of simplicity, since the value of B (this can be easily shown) exerts no influence on the values of the jet parameters.

We now evaluate the magnitude of pressure in the internal region. From the condition of equality of the normal stresses on the separation boundary, without taking into account the capillary forces

$$-p_1 + 2\mu_1 \frac{1+y_*'^2}{1-y_*'^2} \frac{\partial v_1}{\partial y}\bigg|_{y_*} = -p_2 + 2\mu_2 \frac{1+y_*'^2}{1-y_*'^2} \frac{\partial v_2}{\partial y}\bigg|_{y_*} \tag{1.8}$$

we can obtain

$$p_1 = -\rho_2 g x + \frac{2}{5} \frac{\nu_1}{a_0^2} (\mu_1 - \mu_2) A X^{-4/5}.$$

Using the solutions obtained, we write out the expressions for the velocities on the jet axis and the expressions for the velocity and friction on the separation boundary:

$$u_m/U = X^{1/5}[1 + (1/6 - a_3/a_0)X^{-3/5}]; \tag{1.9}$$

$$u_*/U = X^{1/5}[1 - (1/3 + a_3/a_0)X^{-3/5}];$$

$$\tau_* = -\mu_1(U/a_0)X^{-1/5}[1 + (a_3/a_0 - A/5)X^{-3/5}]. \tag{1.10}$$

From the expressions (1.9), (1.10) we see the difference between a jet in the case of viscous interaction between it and the surrounding medium and a jet of ideal liquid, for which $u/U \sim X^{1/2}$.

2. Axisymmetric Jet. In this case, to Eqs. (1.1) we add the equation of the boundary layer in the region 2:

$$\frac{\partial\psi_2}{\partial r}\left(\frac{\partial^2\psi_2}{\partial r\partial x} - \frac{y_*'}{y_*}\frac{\partial\psi_2}{\partial r}\right) - \frac{\partial\psi_2}{\partial x}\frac{\partial^2\psi_2}{\partial r^2} = \nu_2 \frac{\partial^3\psi_2}{\partial r^3}, \tag{2.1}$$

where y_* is the separation surface; $r = y - y_*$; and ψ_2 is the stream function in the axisymmetric flow, i.e., $u_2 = (1/y_*)\partial\psi_2/\partial r$, $v_2 = -(1/y_*)\partial\psi_2/\partial x$. In contrast to the plane case, when considering problems in an axisymmetric formulation we have to take into account the fact that Eq. (2.1) is valid when the thickness of the boundary layer is considerably less than the radius of the internal region. After finding the solutions we can write the limiting condition of applicability of the equation to be used and the solutions obtained below. Carrying out all the reasoning of the preceding case, we can obtain the first terms of the expansions of the stream functions in the internal and external regions. Not dwelling on the details, we write out at once the form of solutions in the first and second regions:

$$\psi_1 = a_0^2 U [F_0(n) + X^{-1/8}F_1(n) + \ldots],$$

$$n = \frac{yX^{1/8}}{a_0 + a_1X^{-1/8} + \ldots};$$

$$\psi_2 = a_0^2 U X^{1/2}[G_0(h) + X^{-1/8}G_1(h) + \ldots],$$

$$h = \frac{r}{X^{3/8}(a_0 + a_1X^{-1/8} + \ldots)}.$$

In order to eliminate the occurrence of logarithmic terms we must put $2a_6 + a_3^2/a_0 = 0$. The boundary conditions on the axis of the jet and at infinity have the form

$$F_j(0) = 0,\ F_j'(0) = 0,\ G_j'(\infty) = 0.$$

On the boundary separating the two liquids (n=1), within the framework of the number of approximations considered, the conditions

$$G_j'(0) = F_j'(1),\ G_j''(0) = \gamma[F_{j+4}''(1) - F_{j+4}'(1)]$$

are fulfilled. After substitution of the expansions into the system (1.1), we obtain equations in the internal region. Without writing them out, we at once give the solutions of these equations:

$$F_0 = (1/2)n^2,\ F_1 = 0,\ F_2 = 0,\ F_3 = 0,$$

$$F_4 = (B/16)n^2(1 - n^2),\ F_5 = 0,\ F_6 = 0,\ F_7 = 0,$$

$$F_8 = (1/16)(4Ba_4/a_0 - A/4)n^2(1 - n^2),\ F_9 = 0,\ F_{10} = 0,$$

where $a_1 = a_2 = a_3 = a_5 = 0$. Taking into account the last equation, we write out the system of equations in the external region, having first replaced $G_j = Cg_j(t)$, $t = Dh$, where $C = (\varkappa A)^{-1/2}$, $D = (\varkappa A)^{1/2}$, $\varkappa = \nu_1/\nu_2$:

$$g_0''' - \frac{1}{4}\left(g_0'^2 - 2g_0 g_0''\right) = 0, \tag{2.2}$$

$$g_0(0) = 0,\ g_0'(0) = 1,\ g_0'(\infty) = 0,\ g_0''(0) = -\frac{1}{2}\gamma D^{-1}B;$$

$$g_4''' + \frac{1}{2}g_0 g_4'' = \frac{a_4}{a_0}g_0'^2, \tag{2.3}$$

$$g_4(0) = \frac{1}{2}C^{-1},\ g_4'(0) = -\frac{1}{4}B,\ g_4'(\infty) = 0,$$

$$g_4''(0) = \frac{1}{2}\gamma D^{-1}\left(\frac{A}{4} - 4\frac{a_4}{a_0}B\right).$$

In the same way as in the preceding problem, the boundary conditions written out ensure the continuity of the physical parameters indicated. In the expressions (2.2), (2.3) we do not write out the equations for $g_1, g_2, g_3, g_5,$ and g_6, since their solutions are zero. The solution of Eq. (2.2), found numerically, is shown in Fig. 3. Using it, we can find

$$A = \varkappa^{-1}\gamma^2 B^2/4(0.587)^2.$$

The function g_4 will be represented in the form of the sum of the solutions

$$g_4 = C_4 g_0' + D_4 + \frac{a_4}{a_0}g_{41},$$

where g_{41}, shown in Fig. 3, was determined from the solution of the nonhomogeneous equation under the following boundary conditions: $g_{41}(0) = 0$, $g_{41}'(0) = 0$, $g_{41}'(\infty) = 0$. From the solution obtained and the appropriate boundary conditions we shall determine the constants

$$C_4 = 0.213\,B,\ D_4 = C^{-1}/2 - C_4,\ a_4/a_0 = -0.062\left(1/2 - \varkappa^{-1}\gamma^2\right)B.$$

Having written out the integral expression for the impulse

$$I = \int_0^{y_*}\rho_1 y u_1^2 dy + \int_{y_*}^{\infty}\rho_2 y_* u_2^2 dy$$

and substituted here the solution thus obtained, we have

$$I = \rho_2 a_0^2 U^2 X^{6/8}\left\{\int_0^\infty G_0'^2 dh + X^{-4/8}\int_0^\infty\left(2G_0'G_4' - \frac{a_4}{a_0}G_0'^2\right)dh\right\} + \frac{1}{2}\rho_1 a_0^2 U^2 X^{2/8}.$$

The magnitude of this impulse is equal to the resultant mass force, i.e.,

$$I = \rho_1 B\frac{U^2}{A}a_0^2\left(\frac{2}{3}X^{6/8} + 4\frac{a_4}{a_0}X^{2/8}\right).$$

Using the condition of conservation of the mass of the outflowing liquid

$$G = \int_0^{y_*}\rho_1 y u_1 dy = \text{const}$$

and having put $B = 1$, we obtain

$$U = \left[2g\frac{G}{\nu_1\rho_1}\frac{\rho_2 - \rho_1}{\rho_1}\right]^{1/2},\quad a_0 = \left(2\frac{G}{\rho_1}\right)^{1/2}\left[2g\frac{G}{\nu_1\rho_1}\frac{\rho_2 - \rho_1}{\rho_1}\right]^{-1/4}.$$

The evaluation of the magnitude of pressure here can also be carried out by means of the expression (1.8), since the stresses corresponding to the axes x and y in the cylindrical coordinate system coincide in form with the stresses in the plane problem [7]. After substitution of the solutions into (1.8) we have

$$p_1 = -\rho_2 g x + \frac{1}{4}\frac{\nu_1}{a_0^2}(\mu_1 - \mu_2)AX^{-6/8}.$$

We shall now obtain a condition under which the use of Eq. (2.1) is valid. From Fig. 3 we see that for $t = 8$ the function g_0' practically can be considered equal to zero. Taking at this point the quantity $r_* = 8D^{-1}X^{3/8}a_0$ for the thickness of the boundary layer in the external jet, from the condition $r_*/y_* \ll 1$ we obtain $X \ll D^2/64$. Taking into account the asymptotic character of the solution, we obtain

$$1 < X \ll \gamma^2/88.2.$$

We write out the expressions for the velocity on the axis of the jet:

$$u_m/U = X^{2/8}[1 + (1/8 - 2a_4/a_0)X^{-4/8}],$$

and also for the velocity and friction on the separation boundary:

$$u_*/U = X^{2/8}[1 - (1/8 + 2a_4/a_0)X^{-4/8}],$$
$$\tau_* = -\mu_1(U/a_0)X^{-1/8}(1/2)[1 + (a_4/a_0 - A/4)X^{-4/8}].$$

In the conclusion, we point out that for an outflow of a heavy liquid downward the solutions written out will be valid for $\rho_1 > \rho_2$ and

$$B = g\frac{a_0^2}{\nu_1 U}\frac{\rho_1 - \rho_2}{\rho_1}.$$

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STABILIZATION OF SOLUTIONS OF TWO-DIMENSIONAL EQUATIONS OF DYNAMICS OF AN IDEAL LIQUID

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UDC 532.5 +517.9

Problems of solvability of initial- and boundary-value problems for two-dimensional nonstationary Euler equations of dynamics of an ideal liquid have been studied by many authors. A review and the corresponding references can be found, for example, in [1, 2]. However, the problem of asymptotic behavior of the solutions of the Euler equation as $t \to \infty$ has not been investigated.

This is apparently explained by the fact that the corresponding boundary-value problems for a stationary Euler equation do not possess the uniqueness property of the solution. In addition, examples exist where a stationary boundary-value problem has a continuum of solutions, as, for example, the problem with the condition of no leakage of the liquid through the boundary of a region of flow. To obtain any results about the asymptotic behavior in the case of $t \to \infty$ of the solutions of nonstationary initial-value problems, we have to single out a class in which the corresponding stationary problem has a unique solution (or a finite number of solutions). One such class was introduced in [3]. The simplest representative of this class is motion without vortices. In the present paper we present sufficient conditions under which the solutions of two-dimensional Euler equations as $t \to \infty$ tend to a potential flow.

Vladivostok. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 85–92, March–April, 1977. Original article submitted March 26, 1976.